Math 381 Complex Variables and Transforms

Instructor: Matthias Nagel due in class 13:35, Feb 15

Homework 3

Exercise 3.1. For two complex numbers $a, b \in \mathbb{C}$, prove the inequality

$$|a+b| \leq |a| + |b|.$$

Exercise 3.2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $|f(z)| \leq \frac{C}{|z|+1}$ for a constant $C > 0$. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$.
Hint: Apply the Cauchy Integral formula for larger and larger circles and estimate.

Exercise 3.3. Let $U \subset \mathbb{C}$ be a simply-connected subset and $f\colon U \to \mathbb{C}$ a holomorphic function which is nowhere zero. Construct a holomorphic function $G\colon U \to \mathbb{C}$ such that $f(z) = \exp(G(z))$, i.e. define $\log f$.
Hint: You can't simply take the composition $\log f$ as the image of the function f might not be contained in a simply-connected subset of $\mathbb{C}$, but you probably know the derivative of G. Now reinvent the wheel.

Exercise 3.4. For the path $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(t) := \exp(t^2 - t) + 2\pi i t$ calculate

$$\int_\gamma z \sinh(z) dz.$$

Exercise 3.5. Let $\widetilde{\log}$ be the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ centered at -1 and based at $i\pi$. We denote the principal branch of the logarithm by log. Compute $f\colon \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ where

$$f(z) := \widetilde{\log} z - \log z.$$

Exercise 3.6. Prove that $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is star-shaped with center 1.